Stability Analysis of Systems with Time-Varying Delay via Relaxed Integral Inequalities

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Abstract

This paper investigates the stability of linear systems with a time-varying delay. The key problem concerned is how to effectively estimate single integral term with time-varying delay information appearing in the derivative of Lyapunov-Krasovskii functional. Two novel integral inequalities are developed in this paper for this estimation task. Compared with the frequently used inequalities based on the combination of Wirtinger-based inequality (or Auxiliary function-based inequality) and reciprocally convex lemma, the proposed ones can provide smaller bounding gap without requiring any extra slack matrix. Four stability criteria are established by applying those inequalities. Based on three numerical examples, the advantages of the proposed inequalities are illustrated through the comparison of maximal admissible delay bounds provided by different criteria.

Keywords: Time-delay system, time-varying delay, stability, relaxed integral inequality, linear matrix inequality

1. Introduction

Time-varying delays are frequently introduced into control loops during implementing of practical control systems through communication networks [1]. The stability, as the basic requirement of control systems, may be destroyed due to the presence of time delays. Hence, the stability analysis of systems with time-varying delays has been becoming a hot topic in the past few decades [2, 3, 4, 5, 6].

The important objective of stability analysis is to find the maximal admissible delay region such that time-delay system remains stable for the time-varying delay within this region [7]. The determination of such region requires suitable stability criteria. Benefit from the advantages of wide applications and easy extension of Lyapunov-Krasovskii functional (LKF) method and the convenient tractability of the linear matrix inequality (LMI), the delay-dependent stability criterion derived in the framework of the LKF and the LMI is the most effective criterion to provide admissible region of the time-varying delay [8].

In order to obtain delay-dependent criteria via the LKF method, the following double integral term is usually applied in the LKF [9]:

$$V_r(t) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)dsd\theta \tag{1}$$

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where $R > 0$ is the Lyapunov matrix to be determined, h is upper bound of time-varying delay (Note that this paper discusses the time-varying delay with zero low bound, i.e., $0 \leq d(t) \leq h$), and $x(t)$ is the system state. Then its derivative includes the following single integral terms with time-varying delay information:

$$\mathcal{S}(t) := -\int_{t-d(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds \tag{2}$$

In order to obtain the LMI-based criterion, a challenging problem is how to find the upper bound of $\mathcal{S}(t)$ [9].

Before 2004, model transformations, together with Park or Moon inequality [10], were generally applied to handle $\mathcal{S}(t)$ [11, 12]. The model transformation may result in additional dynamics and the inequality-based cross term bounding leads to conservatism [13]. The free-weighting-matrix (FWM) approach was proposed in 2004to overcome those drawbacks [14, 15]. However, the second single integral term, $\int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds$, was ignored based on the above methods. Later, the improved FWM approaches [16, 17, 18] without ignoring such term were developed and used to be the most popular method for studying of different time-delay systems [19, 20, 21]. However, the drawback of the FWM-based method is that many slack matrices bring heavy computation complexity, and it is a bit difficult to judge how to introduce slack matrices reasonably [22].

An alternative type of method that estimates $\mathcal{S}(t)$ using bounding inequalities is applied to avoid introducing too many slack matrices. The estimation of $\mathcal{S}(t)$ based on this type of method includes two key steps, namely, 1) two integral terms in $\mathcal{S}(t)$ are estimated respectively via suitable bounding inequalities; and 2) the $d(t)$ with the form, $\frac{1}{d(t)}$ and $\frac{1}{h-d(t)}$, appearing in the transformed quadratic terms is handled via suitable techniques. For the first step, Jensen inequality [23] is commonly used in the early researches. Later, some tighter inequalities, such as Wirtinger-based inequality [9] and auxiliary function based inequality [24], are developed to improve the results. Recently, Bessel-Legendre inequality, which contains the above ones as spacial cases, further increases the estimation accuracy [25]. For the second step, the simplest treatment is to directly replace $d(t)$ with its bounds [26], while the enlargement leads to obvious conservatism. Another way for this task is to use the convex combination method [27] after moving the $d(t)$ in the denominator to the numerator of the quadratic terms via some FWM-based inequalities [8, 28], simple enlargement treatment [29, 36], and vector-redefined method [30], but it usually requires the introducing of many slack matrices and/or the enlargement treatment. The reciprocally convex lemma [31] directly handling the $d(t)$ in the denominator is the most effective method since it leads to least conservatism while only introduces a few slack matrices.

Due to the characteristic of few slack matrices introducing and small conservatism, the combination of the bounding inequality and the reciprocally convex lemma is becoming the most popular framework for estimating $\mathcal{S}(t)$ during the investigation of the systems with time-varying delay. To the best knowledge of the authors, most current researches following this framework still focus on the development of new bounding inequalities for the aforementioned first step task [32, 33, 34, 35]. However, there is no reported research that discusses the tighter estimation of $\mathcal{S}(t)$ considering two steps together. This motivates the present research.

This paper develops two relaxed integral inequalities to estimate $\mathcal{S}(t)$ by considering two integral terms together, instead of the two-step estimation method applied in the existing work. The first (or second) proposed inequality is tighter than the one, obtained via the combination of the Wirtinger-based inequality (or the auxiliary function based inequality) and the reciprocally convex lemma, without requiring any extra slack matrix. Four stability criteria of a linear system with a time-varying delay are established by applying those inequalities. Finally, three numerical examples are given to illustrate the effective of the proposed inequalities and the corresponding criteria.

The reminder of paper is organized as follows. Section 2 gives problem formulation and preliminaries. In Section 3, two novel inequalities are given and the comparison with the commonly used ones is discussed. Section 4 gives several new stability criteria of a linear system with a time-varying delay. Section 5 illustrates the advantages of the proposed method via numerical examples. Conclusions are given in Section 6.

Notations: Throughout this paper, the superscripts T and -1 mean the transpose and the inverse of a matrix, respectively; $\mathcal{R}^n$ denotes the n-dimensional Euclidean space; $\|\cdot\|$ refers to the Euclidean vector norm; $P > 0$ (≥ 0) means P is a real symmetric and positive-definite (semi-positive-definite) matrix; I and 0 stand for the identity matrix and the zero-matrix, respectively; $diag\{\cdot\}$ denotes the block-diagonal matrix; and symmetric term in the symmetric matrix is denoted by $*$.

2. Problem Formulation and Preliminaries

Consider the following linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t-d(t)),\ t \geq 0 \\ x(t) = \phi(t), \quad t \in [-h, 0] \end{cases} \tag{3}$$

where $x(t) \in \mathcal{R}^n$ is the system state, A and A_d are the system matrices, the initial condition $\phi(t)$ is a continuously differentiable function, and $d(t)$ is the time-varying delay satisfying

$$0 \leq d(t) \leq h \tag{4}$$

and

$$\mu_1 \leq \dot{d}(t) \leq \mu_2 \tag{5}$$

where h, μ_1, and μ_2 are constant.

This paper aims to derive new delay-dependent stability criteria for analyzing the stability of system (3). In this paper, the key problem to be concerned during the criterion-deriving is how to estimate the following single integral term with time-varying delay information:

$$\mathcal{S}(t) = -\int_{t-d(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds - \int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds \tag{6}$$

This paper will develop two new inequalities for the above estimation task.

The Wirtinger-based integral inequality [9] and the auxiliary function based inequality [24] to be used are given in the following lemma, shown as inequalities (7) and (8), respectively.

Lemma 1. *[9, 24] For symmetric matrix $R > 0$, scalars a and b with $a < b$, and vector ω such that the integration concerned are well defined, the following inequalities hold*

$$(b-a)\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2 \tag{7}$$

$$(b-a)\int_a^b \dot{\omega}^T(s)R\dot{\omega}(s)ds \geq \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2 + 5\chi_3^T R\chi_3 \tag{8}$$

where

$$\begin{aligned}
\chi_1 &= \omega(b) - \omega(a)\\
\chi_2 &= \omega(b) + \omega(a) - \frac{2}{b-a}\int_a^b \omega(s)ds\\
\chi_3 &= \omega(b) - \omega(a) + \frac{6}{b-a}\int_a^b \omega(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_s^b \omega(u)duds
\end{aligned}$$

The reciprocally convex lemma proposed in [31] is reformulated as the following simple form [9].

Lemma 2. *([31, 9]) For vectors β_1 and β_2, real scalar $\alpha \in (0,1)$, symmetric matrix $R > 0$, and any matrix S satisfying $\begin{bmatrix} R & S \\ * & R \end{bmatrix} \geq 0$, the following inequality holds*

$$\frac{1}{\alpha}\beta_1^T R\beta_1 + \frac{1}{1-\alpha}\beta_2^T R\beta_2 \geq \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}^T \begin{bmatrix} R & S \\ * & R \end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} \tag{9}$$

3. New inequalities for estimating $\mathcal{S}(t)$

This section discusses the methods of estimating $\mathcal{S}(t)$. The commonly used method based on the bounding inequality and the reciprocally convex lemma is reviewed and two inequalities are summarized following the two-step estimation procedure as mentioned in Section I. Then two relaxed inequalities are developed by directly considering two parts of $\mathcal{S}(t)$ together and their advantages compared with the existing ones are briefly discussed.

Firstly, by combining the Wirtinger-based inequality (7) and the reciprocally convex lemma (9), the following inequality is summarized:

Lemma 3. *For a symmetric matrix $R > 0$ and any matrix S_1 satisfying $\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0$ with $R_1 = diag\{R, 3R\}$, the $\mathcal{S}(t)$ defined in (6) can be estimated as:*

$$\mathcal{S}(t) \leq -\frac{1}{h}\zeta_1^T(t)\begin{bmatrix}E_1\\ E_2\end{bmatrix}^T \begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix}\begin{bmatrix}E_1\\ E_2\end{bmatrix}\zeta_1(t) \tag{10}$$

where

$$\begin{aligned}
\zeta_1(t) &= [x^T(t), x^T(t-d(t)), x^T(t-h), v_1^T(t), v_2^T(t)]^T \qquad (11)\\
E_1 &= \begin{bmatrix} \bar{e}_1 - \bar{e}_2 \\ \bar{e}_1 + \bar{e}_2 - 2\bar{e}_4 \end{bmatrix}\\
E_2 &= \begin{bmatrix} \bar{e}_2 - \bar{e}_3 \\ \bar{e}_2 + \bar{e}_3 - 2\bar{e}_5 \end{bmatrix}\\
\bar{e}_i &= [0_{n\times(i-1)n}, I, 0_{n\times(5-i)n}], i = 1, 2, \cdots, 5\\
v_1(t) &= \int_{t-d(t)}^{t} \frac{x(s)}{d(t)}ds\\
v_2(t) &= \int_{t-h}^{t-d(t)} \frac{x(s)}{h-d(t)}ds
\end{aligned}$$

Proof: By estimating two parts of $\mathcal{S}(t)$ respectively via Writinger-based inequality, combining the obtained terms via the reciprocally convex lemma, and following the same lines as in [9], inequality (10) can be easily obtained. The details are omitted here. ■

Inequality (10) is obtained by following two steps. By considering two parts of $\mathcal{S}(t)$ together, the following inequality can be obtained.

Lemma 4. *For a block symmetric matrix $R_1 = diag\{R, 3R\}$ with $R > 0$ and any matrix S_1, the $\mathcal{S}(t)$ defined in (6) can be estimated as:*

$$\mathcal{S}(t) \leq -\frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \left(\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} + \begin{bmatrix} \frac{h-d(t)}{h}T_1 & 0 \\ 0 & \frac{d(t)}{h}T_2 \end{bmatrix}\right)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\zeta_1(t) \tag{12}$$

where $\zeta_1(t)$, E_1, and E_2 are defined in (10), $T_1 = R_1 - S_1 R_1^{-1} S_1^T$, and $T_2 = R_1 - S_1^T R_1^{-1} S_1$.

Proof: By setting $\lambda_1(s, a, b) = \frac{2s-b-a}{b-a}$, the following equations can be obtained via simple calculations [18, 24]:

$$\int_a^b \dot{x}(s)ds = x(b) - x(a) \tag{13}$$

$$\int_a^b \lambda_1(s, a, b)\dot{x}(s)ds = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds \tag{14}$$

$$\int_a^b \lambda_1(s, a, b)ds = 0 \tag{15}$$

$$\int_a^b \lambda_1^2(s, a, b)ds = \frac{b-a}{3} \tag{16}$$

For a symmetric matrix $R > 0$ and any matrices, $M_i, i = 1, 2, 3, 4$, with appropriate dimension, the following holds based on Schur complement:

$$\begin{bmatrix} M_1R^{-1}M_1^T & M_1R^{-1}M_2^T & M_1 \\ * & M_2R^{-1}M_2^T & M_2 \\ * & * & R \end{bmatrix} \geq 0 \tag{17}$$

$$\begin{bmatrix} M_3R^{-1}M_3^T & M_3R^{-1}M_4^T & M_3 \\ * & M_4R^{-1}M_4^T & M_4 \\ * & * & R \end{bmatrix} \geq 0 \tag{18}$$

which lead to

$$\Pi_1 = -\int_{t-d(t)}^{t} \begin{bmatrix} g_1 \\ f_1 g_1 \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} M_1R^{-1}M_1^T & M_1R^{-1}M_2^T & M_1 \\ * & M_2R^{-1}M_2^T & M_2 \\ * & * & R \end{bmatrix} \begin{bmatrix} g_1 \\ f_1 g_1 \\ \dot{x}(s) \end{bmatrix} ds \leq 0 \tag{19}$$

$$\Pi_2 = -\int_{t-h}^{t-d(t)} \begin{bmatrix} g_1 \\ f_2 g_1 \\ \dot{x}(s) \end{bmatrix}^T \begin{bmatrix} M_3R^{-1}M_3^T & M_3R^{-1}M_4^T & M_3 \\ * & M_4R^{-1}M_4^T & M_4 \\ * & * & R \end{bmatrix} \begin{bmatrix} g_1 \\ f_2 g_1 \\ \dot{x}(s) \end{bmatrix} ds \leq 0 \tag{20}$$

where

$$g_1 = [E_1^T, \;\; E_2^T]^T \zeta_1(t), \qquad f_1 = \lambda_1(s, t-d(t), t), \qquad f_2 = \lambda_1(s, t-h, t-d(t))$$

For any matrices, $L_i, i = 1, 2, 3, 4$, with appropriate dimension, define the following notations:

$$\begin{aligned} M_1 &= -\frac{1}{h}\left[R, 0, L_1^T\right]^T, & M_2 &= -\frac{1}{h}\left[0, 3R, L_2^T\right]^T \\ M_3 &= -\frac{1}{h}\left[L_3^T, R, 0\right]^T, & M_4 &= -\frac{1}{h}\left[L_4^T, 0, 3R\right]^T \\ R_1 &= \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix}, & S_1 &= [L_1, L_2]^T = [L_3, L_4] \end{aligned}$$

Carrying out simple algebraic calculation based on (13)-(16) yields

$$
\begin{aligned}
-\int_{t-d(t)}^{t}\begin{bmatrix} g_1\\ f_1g_1\end{bmatrix}^T\begin{bmatrix} M_1R^{-1}M_1^T & M_1R^{-1}M_2^T\\ * & M_2R^{-1}M_2^T\end{bmatrix}\begin{bmatrix} g_1\\ f_1g_1\end{bmatrix}ds &= -\frac{d(t)}{h^2}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left[\begin{array}{cc|c} R & 0 & L_1^T\\ 0 & 3R & L_2^T\\ \hline L_1 & L_2 & L_1R^{-1}L_1^T+L_2(3R)^{-1}L_2^T\end{array}\right]\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t)\\
&= \frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left(\frac{-d(t)}{h}\begin{bmatrix} R_1 & S_1\\ * & S_1^TR_1^{-1}S_1\end{bmatrix}\right)\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t) \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
-2\int_{t-d(t)}^{t}\begin{bmatrix} g_1\\ f_1g_1\end{bmatrix}^T\begin{bmatrix} M_1\\ M_2\end{bmatrix}\dot{x}(s)ds &= \frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left[\begin{array}{cc|c} 2R & 0 & L_1^T\\ 0 & 6R & L_2^T\\ \hline L_1 & L_2 & 0\end{array}\right]\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t)\\
&= \frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\begin{bmatrix} 2R_1 & S_1\\ * & 0\end{bmatrix}\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t) \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
-\int_{t-h}^{t-d(t)}\begin{bmatrix} g_1\\ f_2g_1\end{bmatrix}^T\begin{bmatrix} M_3R^{-1}M_3^T & M_3R^{-1}M_4^T\\ * & M_4R^{-1}M_4^T\end{bmatrix}\begin{bmatrix} g_1\\ f_2g_1\end{bmatrix}ds &= -\frac{h-d(t)}{h^2}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left[\begin{array}{c|cc} L_3R^{-1}L_3^T+L_4(3R)^{-1}L_4^T & L_3 & L_4\\ \hline L_3^T & R & 0\\ L_4^T & 0 & 3R\end{array}\right]\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t)\\
&= \frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left(\frac{d(t)-h}{h}\begin{bmatrix} S_1R_1^{-1}S_1^T & S_1\\ * & R_1\end{bmatrix}\right)\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t) \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
-2\int_{t-h}^{t-d(t)}\begin{bmatrix} g_1\\ f_2g_1\end{bmatrix}^T\begin{bmatrix} M_3\\ M_4\end{bmatrix}\dot{x}(s)ds &= \frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left[\begin{array}{c|cc} 0 & L_3 & L_4\\ \hline L_3^T & 2R & 0\\ L_4^T & 0 & 6R\end{array}\right]\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t)\\
&= \frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\begin{bmatrix} 0 & S_1\\ * & 2R_1\end{bmatrix}\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t) \qquad (24)
\end{aligned}
$$

Using (21)-(24) yields

$$
\Pi_1+\Pi_2 = \mathcal{S}(t)+\frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left(\begin{bmatrix} R_1 & S_1\\ * & R_1\end{bmatrix}+\begin{bmatrix} \frac{h-d(t)}{h}T_1 & 0\\ 0 & \frac{d(t)}{h}T_2\end{bmatrix}\right)\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t) \qquad (25)
$$

The holding of (19) and (20) leads to $\Pi_1+\Pi_2\le 0$. Thus,

$$
\mathcal{S}(t)\le -\frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1\\ E_2\end{bmatrix}^T\left(\begin{bmatrix} R_1 & S_1\\ * & R_1\end{bmatrix}+\begin{bmatrix} \frac{h-d(t)}{h}T_1 & 0\\ 0 & \frac{d(t)}{h}T_2\end{bmatrix}\right)\begin{bmatrix} E_1\\ E_2\end{bmatrix}\zeta_1(t) \qquad (26)
$$

This completes the proof of inequality (12). ■

The relationship between the existing inequality (10) and the proposed inequality (12) is given as the following remark.

Remark 1. *The advantages of the proposed inequality (12), compared with inequality (10), can be shown from the following two aspects:*

1) *On one hand, it is obvious that the slack matrices included in two inequalities are identical, which means that they will introduce the same number of decision variables into the final criteria.*

2) On the other hand, the estimation gaps (calculated by subtracting the left-hand side of inequality from the right-hand side one) of (10) and (12) are respectively denoted by J_1 and J_2, then the following holds

$$J_1 - J_2 = \frac{1}{h}\zeta_1^T(t)\begin{bmatrix}E_1\\E_2\end{bmatrix}^T\begin{bmatrix}\frac{h-d(t)}{h}T_1 & 0\\ 0 & \frac{d(t)}{h}T_2\end{bmatrix}\begin{bmatrix}E_1\\E_2\end{bmatrix}\zeta_1(t) \tag{27}$$

Based on Schur complement, the holding of $\begin{bmatrix}R_1 & S_1\\ * & R_1\end{bmatrix} \geq 0$ leads to $T_1 \geq 0$ and $T_2 \geq 0$. Thus, $J_1 - J_2 \geq 0$, which means inequality (12) provides a closer estimated value of $\mathcal{S}(t)$ and has less conservatism.

Therefore, compared with inequality (10), the proposed inequality (12) has potential to lead to a criterion with less conservatism but without requiring any extra decision variable.

Secondly, by combining the auxiliary function based inequality (8) and the reciprocally convex lemma (9), the following inequality is summarized:

Lemma 5. *For a symmetric matrix $R > 0$ and any matrix S_2 satisfying $\begin{bmatrix}R_2 & S_2\\ * & R_2\end{bmatrix} \geq 0$ with $R_2 = diag\{R, 3R, 5R\}$, the $\mathcal{S}(t)$ defined in (6) can be estimated as:*

$$\mathcal{S}(t) \leq -\frac{1}{h}\zeta_2^T(t)\begin{bmatrix}E_3\\E_4\end{bmatrix}^T\begin{bmatrix}R_2 & S_2\\ * & R_2\end{bmatrix}\begin{bmatrix}E_3\\E_4\end{bmatrix}\zeta_2(t) \tag{28}$$

where

$$\zeta_2(t) = [x^T(t), x^T(t-d(t)), x^T(t-h), v_1^T(t), v_2^T(t), v_3^T(t), v_4^T(t)]^T \tag{29}$$

$$E_3 = \begin{bmatrix} e_1 - e_2\\ e_1 + e_2 - 2e_4\\ e_1 - e_2 + 6e_4 - 12e_6\end{bmatrix}$$

$$E_4 = \begin{bmatrix} e_2 - e_3\\ e_2 + e_3 - 2e_5\\ e_2 - e_3 + 6e_5 - 12e_7\end{bmatrix}$$

$$e_i = [0_{n\times(i-1)n}, I, 0_{n\times(7-i)n}], i = 1, 2, \cdots, 7$$

$$v_1(t) = \int_{t-d(t)}^{t}\frac{x(s)}{d(t)}ds \tag{30}$$

$$v_2(t) = \int_{t-h}^{t-d(t)}\frac{x(s)}{h-d(t)}ds \tag{31}$$

$$v_3(t) = \int_{t-d(t)}^{t}\int_{s}^{t}\frac{x(u)}{d^2(t)}duds \tag{32}$$

$$v_4(t) = \int_{t-h}^{t-d(t)}\int_{s}^{t-d(t)}\frac{x(u)}{(h-d(t))^2}duds \tag{33}$$

Proof: Inequality (28) can be easily obtained by using the auxiliary function based inequality (8) and Lemma 2 and following the same lines as in [9]. The details are omitted here. ■

Similar to Lemma 4, by considering two parts of $\mathcal{S}(t)$ together, the following inequality can be obtained.

Lemma 6. *For a block symmetric matrix $R_2 = diag\{R, 3R, 5R\}$ with $R > 0$ and any matrix S_2, the $\mathcal{S}(t)$ defined in (6) can be estimated as:*

$$\mathcal{S}(t) \leq -\frac{1}{h}\zeta_2^T(t)\begin{bmatrix}E_3\\E_4\end{bmatrix}^T\left(\begin{bmatrix}R_2 & S_2\\ * & R_2\end{bmatrix} + \begin{bmatrix}\frac{h-d(t)}{h}T_3 & 0\\ 0 & \frac{d(t)}{h}T_4\end{bmatrix}\right)\begin{bmatrix}E_3\\E_4\end{bmatrix}\zeta_2(t) \tag{34}$$

where $\zeta_2(t)$, E_3, and E_4 are defined in (28), $T_3 = R_2 - S_2^T R_2^{-1} S_2$, and $T_4 = R_2 - S_2 R_2^{-1} S_2^T$.

Proof: By setting $\lambda_1(s,a,b) = \frac{2s-b-a}{b-a}$ and $\lambda_2(s,a,b) = \frac{6s^2-6(a+b)s+b^2+4ab+a^2}{(b-a)^2}$, the following equations can be obtained based on the integral by parts of calculus [24, 32]:

$$\int_a^b \lambda_2(s,a,b)\dot{x}(s)ds = x(b) - x(a) + \frac{6}{b-a}\int_a^b x(s)ds - \frac{12}{(b-a)^2}\int_a^b\int_s^b x(u)duds \tag{35}$$

$$\int_a^b \lambda_2^2(s,a,b)ds = \frac{b-a}{5} \tag{36}$$

$$\int_a^b \lambda_1(s,a,b)\lambda_2(s,a,b)ds = 0 \tag{37}$$

$$\int_a^b \lambda_2(s,a,b)ds = 0 \tag{38}$$

For any matrices, $L_i, i = 5, 6, \cdots, 10$, with appropriate dimension, define the following notations:

$$g_2 = \left[E_3^T,\ E_4^T\right]^T \zeta_2(t), \qquad S_2 = [L_5, L_6, L_7]^T = [L_8, L_9, L_{10}] \tag{39}$$

$$N_1 = -\frac{1}{h}\left[R, 0, 0, L_5^T\right]^T, \qquad N_2 = -\frac{1}{h}\left[0, 3R, 0, L_6^T\right]^T \tag{40}$$

$$N_3 = -\frac{1}{h}\left[0, 0, 5R, L_7^T\right]^T, \qquad N_4 = -\frac{1}{h}\left[L_8^T, R, 0, 0\right]^T \tag{41}$$

$$N_5 = -\frac{1}{h}\left[L_9^T, 0, 3R, 0\right]^T, \qquad N_6 = -\frac{1}{h}\left[L_{10}^T, 0, 0, 5R\right]^T \tag{42}$$

$$f_3 = \lambda_2(s, t-d(t), t), \qquad f_4 = \lambda_2(s, t-h, t-d(t)) \tag{43}$$

Using (13)-(16), (35)-(43) and following the similar procedure of the proof of inequality (12) yield

$$\begin{aligned}
&\mathcal{S}(t) + \frac{1}{h}\zeta_2^T(t)\begin{bmatrix}E_3\\E_4\end{bmatrix}^T\left(\begin{bmatrix}R_2 & S_2\\ * & R_2\end{bmatrix} + \begin{bmatrix}\frac{h-d(t)}{h}T_3 & 0\\ 0 & \frac{d(t)}{h}T_4\end{bmatrix}\right)\begin{bmatrix}E_3\\E_4\end{bmatrix}\zeta_2(t)\\
&= -\int_{t-d(t)}^{t}\begin{bmatrix}g_2\\f_1g_2\\f_3g_2\\\dot{x}(s)\end{bmatrix}^T\begin{bmatrix}N_1R^{-1}N_1^T & N_1R^{-1}N_2^T & N_1R^{-1}N_3^T & N_1\\ * & N_2R^{-1}N_2^T & N_2R^{-1}N_3^T & N_2\\ * & * & N_3R^{-1}N_3^T & N_3\\ * & * & * & R\end{bmatrix}\begin{bmatrix}g_2\\f_1g_2\\f_3g_2\\\dot{x}(s)\end{bmatrix}ds\\
&\quad -\int_{t-h}^{t-d(t)}\begin{bmatrix}g_2\\f_2g_2\\f_4g_2\\\dot{x}(s)\end{bmatrix}^T\begin{bmatrix}N_4R^{-1}N_4^T & N_4R^{-1}N_5^T & N_4R^{-1}N_6^T & N_4\\ * & N_5R^{-1}N_5^T & N_5R^{-1}N_6^T & N_5\\ * & * & N_6R^{-1}N_6^T & N_6\\ * & * & * & R\end{bmatrix}\begin{bmatrix}g_2\\f_2g_2\\f_4g_2\\\dot{x}(s)\end{bmatrix}ds\\
&\leq 0
\end{aligned} \tag{44}$$

Therefore, inequality (34) can be obtained. ■

Remark 2. *Similar to the discussion shown in Remark 1, it can be find that, compared with inequality (28), the proposed inequality (34) has potential to lead to a criterion with less conservatism but without requiring any extra decision variable.*

Remark 3. *It is worthy pointing out that the proposed inequalities (12) and (34) are developed for estimating two integral terms with time-varying delay information, i.e.,* $\mathcal{S}(t)$*. That is, the advantages of the proposed inequalities can* *be found for studying the system with time-varying delays. For a system with constant delay, the proposed inequalities (12) and (34) will reduce to Wirtinger-based inequality (7) and auxiliary function based inequality (8). Specifically, it can be easily obtained that $E_2 = 0$ for the case of constant delay ($d(t) \equiv h$), then the following holds*

$$\mathcal{S}(t) \leq -\frac{1}{h}\zeta_1^T(t)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \left(\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} + \begin{bmatrix} \frac{h-d(t)}{h}T_1 & 0 \\ 0 & \frac{d(t)}{h}T_2 \end{bmatrix} \right) \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \zeta_1(t) \tag{45}$$

$$= -\frac{1}{h}\zeta_1^T(t)E_1^T R_1 E_1 \zeta_1(t) \tag{46}$$

That is, the proposed inequality (12) reduces to Wirtinger-based inequality (7). Similarly, the proposed inequality (34) reduces to auxiliary function based inequality (8) for the case of constant delay.

Remark 4. *In [25], a Bessel-Legendre inequality is proposed based on Legendre polynomials and Bessel inequality. Considering the Bessel-Legendre inequality with $N = 1$ and $N = 2$ respectively leads to the Wirtinger-based inequality (7) and the auxiliary function based inequality (8). By extending the idea of deriving of inequalities (12) and (34) (i.e., the idea of improving (10) and (28), respectively), a series of new integral inequalities that are tighter than the ones obtained by combining the Bessel-Legendre inequality with $N > 2$ and the reciprocally convex lemma can be developed. Moreover, the proposed inequalities in this paper are applied for the time-varying delay with zero low bound, and the corresponding inequality for the time-varying delay with non-zero low bound can be obtained by following the similar idea. The details are omitted here.*

Remark 5. *Very recently, several Wirtinger-based summation inequalities with the similar form of Wirtinger-based inequality (7) have been developed for discrete-time system with time-varying delay [39, 40, 41, 42]. It is expected that the corresponding tighter summation inequalities can be obtained based on the similar idea of deriving of inequalities (12) and (34).*

4. Application to a linear system with time-varying delay

In this section, the inequalities mentioned in Section 3 are used to derive the stability criteria of system (3). The stability criteria obtained via inequalities (10) and (12), together with their comparison, are given at first. Then, the stability criteria obtained via inequalities (28) and (34), together with their comparison, are discussed.

The stability criteria obtained via inequalities (10) and (12) are summarized as follows.

Theorem 1. *For given scalars h and $\mu_1 \leq 0 \leq \mu_2$, system (3) is asymptotically stable if one of the following conditions holds*

C1: [Derived by (10)] there exist a $3n \times 3n$ matrix $P_1 > 0$, $n \times n$ matrices $Q > 0$, $R > 0$, $Z > 0$, and a $2n \times 2n$ matrix S_1, such that the following LMIs hold for $\dot{d}(t) \in \{\mu_1, \mu_2\}$:

$$\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0 \tag{47}$$

$$\Psi_1 < 0 \tag{48}$$

C2: [Derived by (12)] there exist a $3n \times 3n$ matrix $P_1 > 0$, $n \times n$ matrices $Q > 0$, $R > 0$, $Z > 0$, and a $2n \times 2n$ matrix S_1, such that the following LMIs hold for $\dot{d}(t) \in \{\mu_1, \mu_2\}$:

$$\Phi_1 = \begin{bmatrix} \Psi_2|_{d(t)=0} & E_1^T S_1 \\ * & -R_1 \end{bmatrix} < 0 \tag{49}$$

$$\Phi_2 = \begin{bmatrix} \Psi_2|_{d(t)=h} & E_2^T S_1^T \\ * & -R_1 \end{bmatrix} < 0 \tag{50}$$

where

$$\Psi_1 = \bar{\Xi}_1 + \bar{\Xi}_1^T - \bar{\Xi}_{2a} + \bar{\Xi}_3 \tag{51}$$

$$\Psi_2 = \bar{\Xi}_1 + \bar{\Xi}_1^T - \bar{\Xi}_{2b} + \bar{\Xi}_3 \tag{52}$$

$$\begin{aligned}
\bar{\Xi}_1 &= \mathfrak{E}_1^T P_1 \mathfrak{E}_2 \\
\bar{\Xi}_{2a} &= \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad R_1 = \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \\
\bar{\Xi}_{2b} &= \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} \frac{2h-d(t)}{h} R_1 & S_1 \\ * & \frac{h+d(t)}{h} R_1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \\
\bar{\Xi}_3 &= \bar{e}_1^T (Q+Z) \bar{e}_1 - (1-\dot{d}(t)) \bar{e}_2^T Q \bar{e}_2 - \bar{e}_3^T Z \bar{e}_3 + h^2 \bar{e}_s^T R \bar{e}_s \\
\bar{e}_s &= [A, A_d, 0, 0, 0] \\
\bar{e}_i &= [0_{n\times(i-1)n}, I, 0_{n\times(5-i)n}], i = 1, 2, \cdots, 5 \\
E_i &= \begin{bmatrix} \bar{e}_i - \bar{e}_{i+1} \\ \bar{e}_i + \bar{e}_{i+1} - 2\bar{e}_{i+3} \end{bmatrix}, i = 1, 2 \\
\mathfrak{E}_1 &= \left[\bar{e}_1^T,\ d(t)\bar{e}_4^T,\ (h-d(t))\bar{e}_5^T\right]^T \\
\mathfrak{E}_2 &= \left[\bar{e}_s^T,\ \bar{e}_1^T - (1-\dot{d}(t))\bar{e}_2^T,\ (1-\dot{d}(t))\bar{e}_2^T - \bar{e}_3^T\right]^T
\end{aligned}$$

Proof: Construct the following candidate LKF:

$$V_1(t) = \eta_1^T(t) P_1 \eta_1(t) + \int_{t-d(t)}^{t} x^T(s) Q x(s) ds + \int_{t-h}^{t} x^T(s) Z x(s) ds + h \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta \tag{53}$$

where

$$\eta_1(t) = \left[x^T(t), \int_{t-d(t)}^{t} x^T(s) ds, \int_{t-h}^{t-d(t)} x^T(s) ds \right]^T$$

and $P_1 > 0$, $Q > 0$, $Z > 0$, and $R > 0$. It is easily found that the LKF satisfies $V_1(t) \geq \epsilon_1 \|x(t)\|^2$ with $\epsilon_1 > 0$.

Calculating the derivative of $V_1(t)$ yields

$$\begin{aligned}
\dot{V}_1(t) &= 2\eta_1^T(t) P_1 \dot{\eta}_1(t) + x^T(t)(Q+Z)x(t) - (1-\dot{d}(t)) x^T(t-d(t)) Q x(t-d(t)) - x^T(t-h) Z x(t-h) \\
&\quad + h^2 \dot{x}^T(t) R \dot{x}(t) - h \int_{t-d(t)}^{t} \dot{x}^T(s) R \dot{x}(s) ds - h \int_{t-h}^{t-d(t)} \dot{x}^T(s) R \dot{x}(s) ds \\
&= \zeta_1^T(t)(\bar{\Xi}_1 + \bar{\Xi}_1^T + \bar{\Xi}_3)\zeta_1(t) - h\mathcal{S}(t)
\end{aligned} \tag{54}$$

where $\bar{\Xi}_1$ and $\bar{\Xi}_3$ are defined in (51).

On the one hand, if applying inequality (10) to estimate $\mathcal{S}(t)$ appearing in (54), the $\dot{V}_1(t)$ can be estimated as

$$\begin{aligned}
\dot{V}_1(t) &\leq \zeta_1^T(t) \left\{ \bar{\Xi}_1 + \bar{\Xi}_1^T + \bar{\Xi}_3 - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \right\} \zeta_1(t) \\
&= \zeta_1^T(t) \Psi_1 \zeta_1(t)
\end{aligned} \tag{55}$$

where $\zeta_1(t)$ is defined in (11) and Ψ_1 is defined in (51). Therefore, $\Psi_1 < 0$ leads to $\dot{V}_1(t) \leq -\epsilon_2 \|x(t)\|^2$ for a sufficient small scalar $\epsilon_2 > 0$. Hence, the holding of (47) and (48) ensures the asymptotical stability of system (3). This completes the proof of Theorem 1.C1.

On the other hand, if applying inequality (12) to estimate $\mathcal{S}(t)$ appearing in (54), the $\dot{V}_1(t)$ can be estimated as

$$\begin{aligned}\dot{V}_1(t) \leq & \zeta_1^T(t)\left\{\bar{\Xi}_1 + \bar{\Xi}_1^T + \bar{\Xi}_3 - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \left(\begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} + \begin{bmatrix} \frac{h-d(t)}{h}T_1 & 0 \\ 0 & \frac{d(t)}{h}T_2 \end{bmatrix}\right)\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}\right\}\zeta_1(t) \\ = & \zeta_1^T(t)(\Psi_2 + \Xi_a)\zeta_1(t) \end{aligned} \tag{56}$$

where Ψ_2 is defined in (52), and

$$\Xi_a = \frac{h - d(t)}{h} E_1^T S_1 R_1^{-1} S_1^T E_1 + \frac{d(t)}{h} E_2^T S_1^T R_1^{-1} S_1 E_2 \tag{57}$$

Therefore, $\Phi_i < 0, i = 1, 2$, which is equivalent to $\Psi_2 + \Xi_a < 0$ based on Schur complement and convex combination method, leads to $\dot{V}_1(t) \leq -\epsilon_2 ||x(t)||^2$ for a sufficient small scalar $\epsilon_2 > 0$. Hence, the holding of (49) and (50) ensures the asymptotical stability of system (3). This completes the proof of Theorem 1.C2. ■

Theorem 1.C1 (same to Theorem 7 of [9]) and Theorem 1.C2 are derived by respectively using the existing inequality (10) and the proposed inequality (12) to estimate the $\mathcal{S}(t)$ arising in the derivative of the same LKF. That is, the only difference is that two different inequalities are used to achieve the estimation task. Therefore, the advantage of inequality (12) compared with inequality (10) can be found through the comparison of the results provided by those two criteria. Furthermore, it can be proved that Theorem 1.C2 is less conservative than Theorem 1.C1, as representation in the following theorem.

Theorem 2. *Theorem 1.C2 is less conservative than Theorem 1.C1 for the time-varying delay case ($\mu_i \neq 0$), namely,*

- *When there exist feasible solutions of (47) and (48) for any given scalars h and $\mu_1 \leq 0 \leq \mu_2$, there must exist feasible solutions of (49) and (50) for the same h and $\mu_1 \leq 0 \leq \mu_2$; and*
- *When there does not exist feasible solutions of (47) and (48) for some given scalars h and $\mu_1 < 0 < \mu_2$, there may still exist feasible solutions of (49) and (50) for the same h and $\mu_1 < 0 < \mu_2$.*

That is, for any fixed $\mu_1 \leq 0 \leq \mu_2$, Theorem 1.C2 will provide bigger maximal admissible delay upper bounds, $h_{\max}$, in compared with Theorem 1.C1,

Proof: For the conditions, $\Phi_i < 0, i = 1, 2$, of Theorem 1.C2, the following relationship is true:

$$\Phi_i < 0, i = 1, 2 \Leftrightarrow \Phi = \Psi_1 - \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} \frac{h-d(t)}{h}T_1 & 0 \\ 0 & \frac{d(t)}{h}T_2 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} < 0 \tag{58}$$

On the one hand, for any given scalars h and $\mu_1 \leq 0 \leq \mu_2$, the feasible solutions, (P_1, Q, R, Z, S_1), of (47) and (48) lead to

$$\left.\begin{array}{r} \begin{bmatrix} R_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0 \Rightarrow T_1 \geq 0, T_2 \geq 0 \\ \Psi_1 < 0 \end{array}\right\} \Rightarrow \Phi < 0 \Rightarrow \Phi_i < 0, i = 1, 2 \tag{59}$$

Thus, the matrices (P_1, Q, R, Z, S_1) must be the feasible solutions of (49) and (50).

On the other hand, when there is no feasible solution of (47) and (48) for some given scalars h and $\mu_1 < 0 < \mu_2$. That is, for all possible combinations of matrices (P_1, Q, R, Z, S_1), no one can lead to

$$\Psi_1 < 0 \tag{60}$$

However, for the time-varying delay case ($\mu_i \neq 0$), there may still exist one or more sets of matrices, (P_1, Q, R, Z, S_1), satisfying the following condition

$$\Psi_1 < \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}^T \begin{bmatrix} \frac{h-d(t)}{h} T_1 & 0 \\ 0 & \frac{d(t)}{h} T_2 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{61}$$

which means $\Phi < 0$, thus, $\Phi_i < 0, i = 1, 2$. Therefore, those matrices are the feasible solutions of (49) and (50). This completes the proof. ■

The stability criteria obtained via inequalities (28) and (34) are summarized as follows.

Theorem 3. *For given scalars h and $\mu_1 \leq 0 \leq \mu_2$, system (3) is asymptotically stable if one of the following conditions holds*

C1: [Derived by (28)] there exist a $5n \times 5n$ matrix $P_2 > 0$, $n \times n$ matrices $Q > 0$, $R > 0$, $Z > 0$, and a $3n \times 3n$ matrix S_2 such that the following LMIs hold for $\dot{d}(t) \in \{\mu_1, \mu_2\}$:

$$\begin{bmatrix} R_2 & S_2 \\ * & R_2 \end{bmatrix} \geq 0 \tag{62}$$

$$\Psi_3 < 0 \tag{63}$$

C2: [Derived by (34)] there exist a $5n \times 5n$ matrix $P_2 > 0$, $n \times n$ matrices $Q > 0$, $R > 0$, $Z > 0$, and a $3n \times 3n$ matrix S_2 such that the following LMIs hold for $\dot{d}(t) \in \{\mu_1, \mu_2\}$:

$$\Phi_3 = \begin{bmatrix} \Psi_4|_{d(t)=0} & E_3^T S_2 \\ * & -R_2 \end{bmatrix} < 0 \tag{64}$$

$$\Phi_4 = \begin{bmatrix} \Psi_4|_{d(t)=h} & E_4^T S_2^T \\ * & -R_2 \end{bmatrix} < 0 \tag{65}$$

where

$$\Psi_3 = \Xi_1 + \Xi_1^T - \Xi_{2a} + \Xi_3 \tag{66}$$

$$\Psi_4 = \Xi_1 + \Xi_1^T - \Xi_{2b} + \Xi_3 \tag{67}$$

$$\Xi_1 = \mathfrak{E}_3^T P_2 \mathfrak{E}_4$$

$$\Xi_{2a} = \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}^T \begin{bmatrix} R_2 & S_2 \\ * & R_2 \end{bmatrix} \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}, \quad R_2 = \begin{bmatrix} R & 0 & 0 \\ 0 & 3R & 0 \\ 0 & 0 & 5R \end{bmatrix}$$

$$\Xi_{2b} = \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}^T \begin{bmatrix} \frac{2h-d(t)}{h} R_2 & S_2 \\ * & \frac{h+d(t)}{h} R_2 \end{bmatrix} \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}$$

$$\Xi_3 = e_1^T (Q + Z) e_1 - (1 - \dot{d}(t)) e_2^T Q e_2 - e_3^T Z e_3 + h^2 e_s^T R e_s$$

$$e_s = [A, A_d, 0, 0, 0, 0, 0]$$

$$e_i = [0_{n\times(i-1)n}, I, 0_{n\times(7-i)n}], i = 1, 2, \cdots, 7$$

$$E_{i+2} = \begin{bmatrix} e_i - e_{i+1} \\ e_i + e_{i+1} - 2e_{i+3} \\ e_i - e_{i+1} + 6e_{i+3} - 12e_{i+5} \end{bmatrix}, i = 1, 2$$

$$\mathfrak{E}_3 = \left[e_1^T,\ d(t)e_4^T,\ (h - d(t))e_5^T,\ d(t)e_6^T,\ (h - d(t))e_7^T \right]^T$$

$$\mathfrak{E}_4 = \left[e_s^T,\ e_1^T - (1 - \dot{d}(t))e_2^T,\ (1 - \dot{d}(t))e_2^T - e_3^T,\ e_1^T - e_4^T + \dot{d}(t)(e_4 - e_6)^T,\ e_2^T - e_5^T + \dot{d}(t)(e_7 - e_2)^T \right]^T$$

Proof: Construct the following candidate LKF:

$$V_2(t) = \eta_2^T(t) P_2 \eta_2(t) + \int_{t-d(t)}^{t} x^T(s) Q x(s) ds + \int_{t-h}^{t} x^T(s) Z x(s) ds + h \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) R \dot{x}(s) ds d\theta \tag{68}$$

where

$$\eta_2(t) = \left[x^T(t), \int_{t-d(t)}^{t} x^T(s)ds, \int_{t-h}^{t-d(t)} x^T(s)ds, \int_{t-d(t)}^{t}\int_{s}^{t} \frac{x^T(u)}{d(t)}duds, \int_{t-h}^{t-d(t)}\int_{s}^{t-d(t)} \frac{x^T(u)}{h-d(t)}duds \right]^T$$

and $P_2 > 0$, $Q > 0$, $Z > 0$, and $R > 0$. It is easily found that the LKF satisfies $V_2(t) \geq \epsilon_3 \|x(t)\|^2$ with $\epsilon_3 > 0$.

Calculating the derivative of $V_2(t)$ yields

$$\begin{aligned}\dot{V}_2(t) = & 2\eta_2^T(t)P_2\dot{\eta}_2(t) + x^T(t)(Q+Z)x(t) - (1-\dot{d}(t))x^T(t-d(t))Qx(t-d(t)) - x^T(t-h)Zx(t-h) \\ & + h^2\dot{x}^T(t)R\dot{x}(t) - h\int_{t-d(t)}^{t} \dot{x}^T(s)R\dot{x}(s)ds - h\int_{t-h}^{t-d(t)} \dot{x}^T(s)R\dot{x}(s)ds \\ = & \zeta_2^T(t)(\Xi_1 + \Xi_1^T + \Xi_3)\zeta_2(t) - h\mathcal{S}(t) \end{aligned} \tag{69}$$

where Ξ_1 and Ξ_3 are defined in (66).

On the one hand, if applying inequality (28) to estimate $\mathcal{S}(t)$ appearing in (69), the $\dot{V}_2(t)$ can be estimated as

$$\begin{aligned}\dot{V}_2(t) \leq & \zeta_2^T(t)\left\{ \Xi_1 + \Xi_1^T + \Xi_3 - \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}^T \begin{bmatrix} R_2 & S_2 \\ * & R_2 \end{bmatrix} \begin{bmatrix} E_3 \\ E_4 \end{bmatrix} \right\} \zeta_2(t) \\ = & \zeta_2^T(t)\Psi_3\zeta_2(t) \end{aligned} \tag{70}$$

where $\zeta_2(t)$ is defined in (29) and Ψ_3 is defined in (66). Therefore, $\Psi_3 < 0$ leads to $\dot{V}_2(t) \leq -\epsilon_4\|x(t)\|^2$ for a sufficient small scalar $\epsilon_4 > 0$. Hence, the holding of (62) and (63) ensures the asymptotical stability of system (3). This completes the proof of Theorem 3.C1.

On the other hand, if applying inequality (34) to estimate $\mathcal{S}(t)$ appearing in (69), the $\dot{V}_2(t)$ can be estimated as

$$\begin{aligned}\dot{V}_2(t) \leq & \zeta_2^T(t)\left\{ \Xi_1 + \Xi_1^T + \Xi_3 - \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}^T \left(\begin{bmatrix} R_2 & S_2 \\ * & R_2 \end{bmatrix} + \begin{bmatrix} \frac{h-d(t)}{h}T_3 & 0 \\ 0 & \frac{d(t)}{h}T_4 \end{bmatrix} \right) \begin{bmatrix} E_3 \\ E_4 \end{bmatrix} \right\} \zeta_2(t) \\ = & \zeta_2^T(t)(\Psi_4 + \Xi_b)\zeta_2(t) \end{aligned} \tag{71}$$

where Ψ_4 is defined in (67), and

$$\Xi_b = \frac{h-d(t)}{h} E_3^T S_2 R_2^{-1} S_2^T E_3 + \frac{d(t)}{h} E_4^T S_2^T R_2^{-1} S_2 E_4 \tag{72}$$

Therefore, $\Phi_i < 0, i = 3, 4$, which is equivalent to $\Psi_4 + \Xi_b < 0$ based on Schur complement and convex combination method, leads to $\dot{V}_2(t) \leq -\epsilon_4\|x(t)\|^2$ for a sufficient small scalar $\epsilon_4 > 0$. Hence, the holding of (64) and (65) ensures the asymptotical stability of system (3). This completes the proof of Theorem 3.C2. ■

Theorem 3.C1 and Theorem 3.C2 are derived by respectively using the existing inequality (28) and the proposed inequality (34) to estimate the $\mathcal{S}(t)$ arising in the derivative of the same LKF. That is, the only difference is that two different inequalities are used to achieve the estimation task. Therefore, the advantage of inequality (34) compared with inequality (28) can be found through the comparison of the results provided by those two criteria. Similar to Theorem 2, it can be proved that Theorem 3.C2 is less conservative than Theorem 3.C1, as representation in the following theorem.

Theorem 4. *Theorem 3.C2 is less conservative than Theorem 3.C1 for the time-varying delay case ($\mu_i \neq 0$), namely,*

- *When there exist feasible solutions of (62) and (63) for any given scalars h and $\mu_1 \leq 0 \leq \mu_2$, there must exist feasible solutions of (64) and (65) for the same h and $\mu_1 \leq 0 \leq \mu_2$; and*

- *When there does not exist feasible solutions of (62) and (63) for some given scalars h and $\mu_1 < 0 < \mu_2$, there may still exist feasible solutions of (64) and (65) for the same h and $\mu_1 < 0 < \mu_2$.*

That is, for any fixed $\mu_1 \le 0 \le \mu_2$, Theorem 3.C2 will provide bigger maximal admissible delay upper bounds, $h_{\max}$, in compared with Theorem 3.C1,

Proof: The above theorem can be obtained by following similar procedure of the proof of Theorem 2. ■

Remark 6. *It is easy to find that Theorem 3.C1 (Theorem 3.C2) is less conservative than Theorem 1.C1 (Theorem 1.C2), since the later is included by the former as a special case. In fact, Theorem 3.C1 (Theorem 3.C2) will reduce to Theorem 1.C1 (Theorem 1.C2) by following two steps:*

- *Set* $P_2 = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}$;
- *Delete the columns and the rows with all zero elements in the conditions of Theorem 3.C1 (or Theorem 3.C2).*

On the other side, Theorem 1.C2 and Theorem 3.C1 improve the Theorem 1.C1 by following different ways. However, it cannot be determined which one is less conservative between them. The LKF applied for Theorem 3.C1 ($V_2(t)$) is better than the one for Theorem 1.C2 ($V_1(t)$) due to more augmented vectors included, while, for inequalities (12) and (28) respectively used for Theorem 1.C2 and Theorem 3.C1, it is difficult to judge which inequality is better. In fact, it will show that, in the numerical examples, Theorem 1.C2 may lead to less conservative results for some cases or more conservative results for other cases than Theorem 3.C1 does.

Remark 7. *Except for the vectors, $v_i(t), i = 1, 2, \cdots, 4$, many other state-based vectors, such as $x(t-d(t))$, $x(t-h/2)$, $x(t-h)$, and $\dot{x}(t)$, were used to construct more general form of augmented LKFs in literature [17, 18, 26, 30]. Among those vectors, the time-varying delay based vector, $x(t-d(t))$, introduced into the non-integral term of LKF seems to be very helpful to reduce the conservatism [18]. However, the criterion (see e.g., Theorem 1 of [18]) derived based on such type of LKF is no longer suitable for the system with fast-varying delay or unmeasurable delay changing rate. Therefore, this paper does not derive the criterion via such LKF.*

5. Examples

Three numerical examples listed in Table 1 are used to verify the advantages of the proposed inequalities and the corresponding stability criteria. The conservatism of the criteria is checked based on the calculated maximal admissible delay upper bounds (MADUPs). Moreover, the index of the number of decision variables (NoV) is applied to show the complexity of criteria.

Table 1: Systems used as numerical examples

Examples	System parameters
1	$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-d(t))$
2	$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x(t-d(t))$
3	$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} x(t-d(t))$

The MADUPs with respect to various μ calculated by Theorems 1 and 3, as well the ones reported in some existing literature, are listed in Tables 2-4. The NoVs are also given to compare the computation complexity, where

Table 2: MADUPs for various $\mu = -\mu_1 = \mu_2$ (Example 1).

Methods	$\mu = -\mu_1 = \mu_2$						NoVs
	0	0.1	0.5	0.8	1.0	1000	
Corollary 3 [27]	4.472	3.669	2.337	1.934	1.868	1.868	$31.5n^2 + 7.5n$
Theorem 1 [8]	4.975	3.869	2.337	1.934	1.868	1.868	$49n^2 + 5n$
Theorem 2 [37]	5.120	4.081	2.528	2.152	1.991		$35.5n^2 + 3.5n$
Theorem 3 [38]	6.117	4.794	2.682	1.957	1.602		$NoV_{[38]}$
Corollary 1 [18]	6.059	4.710	2.459	2.212	2.186	2.180	$54n^2 + 9n$
Theorem 7 [9]	6.059	4.703	2.420	2.137	2.128	2.113	$10n^2 + 3n$
Theorem 1.C1	6.059	4.703	2.420	2.137	2.128	2.113	$10n^2 + 3n$
Theorem 1.C2	6.059	4.707	2.428	2.205	2.204	2.205	$10n^2 + 3n$
Theorem 2.C1	6.165	4.713	2.570	2.281	2.232	2.113	$23n^2 + 4n$
Theorem 2.C2	6.165	4.714	2.608	2.375	2.319	2.205	$23n^2 + 4n$

Table 3: MADUPs for various $\mu = -\mu_1 = \mu_2$ (Example 2).

Methods	$\mu = -\mu_1 = \mu_2$						NoVs
	0	0.05	0.10	0.50	3.00	1000	
Corollary 3 [27]	2.52	1.81	1.75	1.61	1.60	1.60	$31.5n^2 + 7.5n$
Theorem 1 [8]	2.523	2.166	2.028	1.622	1.608	1.608	$49n^2 + 5n$
Corollary 1 [18]	3.034	2.553	2.372	1.713	1.634	1.634	$54n^2 + 9n$
Theorem 7 [9]	3.034	2.551	2.369	1.700	1.648	1.648	$10n^2 + 3n$
Theorem 1.C1	3.034	2.551	2.369	1.700	1.648	1.648	$10n^2 + 3n$
Theorem 1.C2	3.034	2.553	2.373	1.706	1.652	1.652	$10n^2 + 3n$
Theorem 2.C1	3.136	2.590	2.386	1.775	1.655	1.648	$23n^2 + 4n$
Theorem 2.C2	3.136	2.598	2.397	1.787	1.665	1.652	$23n^2 + 4n$

Table 4: MADUPs for various $\mu = -\mu_1 = \mu_2$ (Example 3).

Methods	$\mu = -\mu_1 = \mu_2$						NoVs
	0.1	0.5	0.7	1.0	3.0	1000	
Theorem 1 [8]	5.876	1.430	1.239	1.225	1.176	1.139	$49n^2 + 5n$
Theorem 2 [30]	5.57	1.35	1.06	1.06	1.06	1.06	$12n^2 + 4n$
Corollary 1 [18]	6.601	1.549	1.406	1.316	1.206	1.202	$54n^2 + 9n$
Theorem 7 [9]	6.590	1.411	1.300	1.245	1.199	1.196	$10n^2 + 3n$
Theorem 1.C1	6.590	1.411	1.300	1.245	1.199	1.196	$10n^2 + 3n$
Theorem 1.C2	6.602	1.447	1.320	1.256	1.208	1.208	$10n^2 + 3n$
Theorem 2.C1	6.604	1.573	1.387	1.294	1.221	1.196	$23n^2 + 4n$
Theorem 2.C2	6.610	1.687	1.462	1.329	1.223	1.208	$23n^2 + 4n$

$NoV_{[38]} = 2[(1+n_\phi)^2 + (6+2n_\phi)(7+2n_\phi) + 2]n^2 + (3+n_\phi)n > 90n^2 + 3n$ with n being the order of system matrix and $n_\phi > 0$ being the order of the filter system.

Based on the results listed in three tables, three observations can be summarized. Firstly, the advantages of the proposed inequality (12) (or (34)) compared with the existing inequality (10) (or (28)) can be found.

- On one hand, the results show that Theorem 1.C2 provides bigger MADUPs than Theorem 1.C1 (i.e., Theorem 7 of [9]) does, which verifies the less conservatism of inequality (12). Similarly, the less conservatism of inequality (34) compared with inequality (28) is verified based on the comparison of the MADUPs provided by Theorem 3.C1 and Theorem 3.C2.

- On the other hand, the NoV of Theorem 1.C2 (or Theorem 3.C2) is the same as that of Theorem 1.C1 (or Theorem 3.C1), which means that the former improves the results but does not require extra decision variables.

Secondly, the results also show the statements of Remarks 3 and 6.

- For the case of constant delay, $\mu = 0$, Theorem 1.C1 and Theorem 3.C1 lead to the same MADUPs, and Theorem 1.C2 and Theorem 3.C2 also lead to the same MADUPs. It verifies the statement of Remark 3.

- On one hand, compared with Theorem 1.C1 (Theorem 1.C2), Theorem 3.C1 (Theorem 3.C2) can lead to bigger (or the same) MADUPs, which means the former is less conservative. On the other hand, based on the comparison of the results provided by Theorem 1.C2 and Theorem 3.C1, it can be found that Theorem 1.C2 is less conservative for some cases ($\mu = 1000$) but is more conservative for other cases in compared with Theorem 3.C1, which means that it is difficult to directly determine Theorem 1.C2 or Theorem 3.C1 is better. Those observations verify the statement of Remark 6.

Finally, Theorem 2.C2 is less conservative between two criteria derived by using the proposed inequalities (i.e., Theorem 1.C2 and Theorem 2.C2), and its advantages in compared with the existing ones can be found.

- Compared with the criteria obtained by different inequalities (Jensen inequality [27], Wirtinger-based inequality [9], free-matrix-based inequality [18]), the new type of LKF [8], and the augmented system model [37], the proposed Theorem 2.C2 provides bigger MADUPs but requires a smaller NoV.

- Although the NoV of Theorem 2.C2 is bigger than that of Theorem 2 of [30], the MADUPs provided by Theorem 2.C2 are obviously larger than those reported in [30]. Theorem 3 of [38] leads to bigger MADUPs for some cases (see Table 2), but its NoV is greatly bigger than the NoV of Theorem 2.C2.

6. Conclusions

This paper has proposed two novel integral inequalities for the stability analysis of linear systems with a time-varying delay. Compared with two inequalities in literature, obtained by combining the widely used Wirtinger-based inequality (or the recently developed auxiliary function based inequality) and the reciprocally convex lemma, the proposed inequalities reduce the estimation gap arising from the estimation of single integral term with time-varying delay information while does not require any extra slack matrix. Four stability criteria of linear system with a time-varying delay have been established by applying those inequalities. Finally, three numerical examples have been given to verify the advantages of the proposed inequalities and the related stability criteria.

Acknowledgments

This work is supported partially by the National Natural Science Foundation of China under Grant Nos. 61503351, 51428702, and 61573325, and the Hubei Provincial Natural Science Foundation of China under Grant 2015CFA010.

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